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ON 4-VALENT GRAPHS IMBEDDED IN ORIENTABLE 2-MANIFOLDS

by

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1. Introduction. For a 3-connected regular 4-valent graph (without loops and multiple edges) imbedded in a closed connected orientable 2-manifold P let $p_k(M)$ denote the number of k-gonal faces (cells, countries) of the map M defined by the imbedded graph. From Euler's formula ($\sum_{k\geqq 3} p_k(M)+v(M) = h(M)+2(1-g)$, $-v(M)$ or $h(M)$ denotes the number of vertices or edges of the map M respectively, g is the genus of P) follows

$$\sum_{k\geqq 3} (4-k)p_k(M) = 8(1-g) \tag{1}$$

Clearly, (1) does not impose restrictions on the number $p_4(M)$. Thus the following question can be asked: Given a sequence $p=(p_3, p_4, \dots, p_s)$ of non-negative integers satisfying

$$\sum_{k\geqq 3} (4-k)p_k = 8(1-g) \tag{2}$$

does there exist a map M with a 4-valent graph on the orientable surface P_g of genus g such that $p_k(M)=p_k$ for all $k\neq 4$? (If so then the sequence p is called *4-realizable on P_g*.)

For $g=0$ the solution is given in GRÜNBAUM [3, 5], for $g=1$ (as well as $g=0$) in BARNETTE—JUCOVIČ—TRENKLER [1] (cf. ZAKS [9]). The aim of this paper is to present the solution for all finite $g\geqq 3$, and to give a partial answer for $g=2$.

Our main result is contained in

THEOREM 1. *Every sequence $p=(p_3, p_4, \dots, p_s)$ of non-negative integers satisfying (2) with $g\geqq 3$ is 4-realizable on the closed connected orientable 2-manifold of genus g.*

2. PROOF of Theorem 1 will be carried out by constructing, for every sequence p satisfying the assumption, the required map on the appropriate manifold. (We recall that there is, to every $g\geqq 0$, exactly one topological type of an orientable 2-manifold of genus g. For our purposes ist interpretation by a sphere with g "handles" will be convenient.) The construction is decomposed into several stages with two different procedures of construction. Luckily there is no need to consider many cases.

First let us decompose the sequence p into two sequences $r=(r_3, r_5, \dots, r_s)$

and $p'=(p'_3, p'_5, \ldots, p'_s)$ of non-negative integers (p_4, r_4, p'_4 need not to be considered) such that

(i) $p_i = p'_i + r_i$ for all i, and

(ii) $\sum_{k \geqq 3} (4-k) r_k = 8(1-g)$, and

(iii) $p'_3 \neq 1$ holds, and

(iv) there exists no sequence $r'=(r'_3, \ldots, r'_s)$ with $r'_i \leqq r_i$ for all i, different from r, satisfying conditions (i), (ii), (iii).

Notice that a) for the sequence p' holds $\sum_{k \geqq 3} (4-k)p'_k = 0$, b) from (ii) follows $\sum_{i \geqq 5} r_i$ to be even for i odd, if $r_3=0$.

The stages of our construction, in rough outline, are:

I. Constructing on the surface of the sphere a map R with r_i i-gonal faces and $2g$ "other" faces to be used as openings for forming the handles. Only the vertices of these last polygons are trivalent, all other vertices are 4-valent.

II. Constructing, on a tube, a map with p'_i i-gonal faces for all i. On the openings of the tube there are equal numbers of vertices and they are all 3-valent. All the other vertices are 4-valent.

III. Joining the tube from II. with the sphere in I. and adding $(g-1)$ handles decomposed into quadrangles.

I. The map R will be obtained from that on Fig. 1 consisting of one $(8g-4)$-gon $A_1 A_2 \ldots A_{4g-3} A_{4g-2} B_{4g-2} \ldots B_2 B_1 = \mathsf{A}$, two triangles $C_1 D_1 E_1$, $C_{2g} D_{2g} E_{2g}$ (the outer handle-openings), $2g-2$ hexagons $C_i D_i E_i Z_i V_i U_i$, $i = 2, \ldots, 2g-1$ (the inner handle openings) and $16g-8$ quadrangles. (We designate by 0_i that handle-opening whose vertices have indices i.) In the sequel the face-aggregate obtained by dissecting the polygon A by new edges will be called *submap A*; the remaining part of the map in Fig. 1 as well as its transforms will be called *submap B*.

The vertices C_i, D_i, E_i, U_i, V_i, Z_i are trivalent, all other vertices are 4-valent. The r_k k-gons, $k \geqq 5$, are cut off from the polygon A, the triangles are inserted in the submap B.

Consider first the case when

α) $r_3 = 0$.

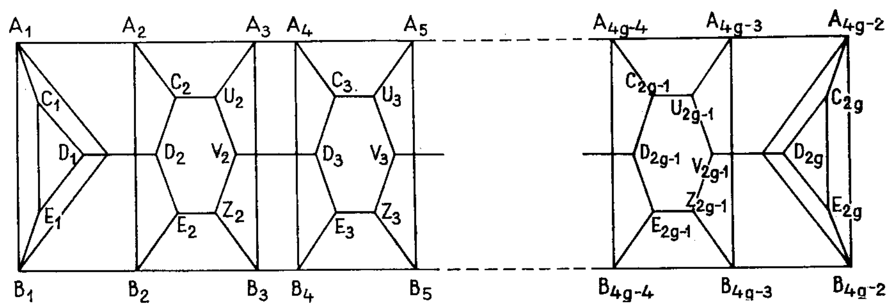

Fig. 1

If $k=2m$ choose a point Q_1 between A_{m-1} and A_m, and a point Q_2 between B_{m-1} and B_m. Joining Q_1 and Q_2 by an edge a k-gon is cut off from the polygon A. The vertices Q_1, Q_2 are still trivalent, however only vertices on the "handle openings" can remain so. If the line Q_1Q_2 does not meet such an opening we draw it and create quadrangles and 4-valent vertices in submap B only. In the opposite case the point Q_1 is joined with a point Q_3 between the points C_j and U_j, and the point Q_2 likewise with the point Q_4 lying on the edge E_jZ_j, $j=\frac{m+1}{2}$. (See Fig. 2). Analogously any further polygon with an even number of edges is cut off from the polygon A.

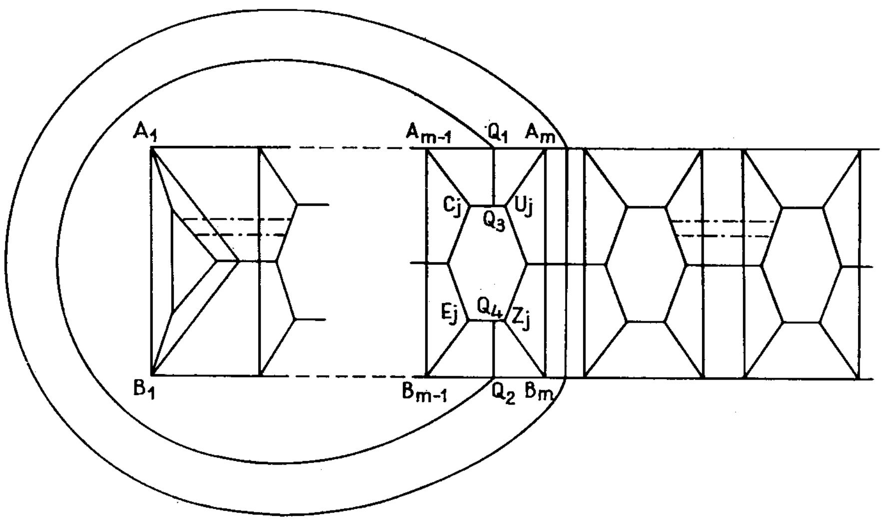

Fig. 2

Faces with odd numbers of edges are cut off in pairs. So if a k-gon, $k=2m+1$, and a t-gon, $t=2n+1$, are needed first a $2(m+n-1)$-gon is cut off, by an edge PQ, the point P lying between the points B_{m+n-2} and B_{m+n-1}, the point Q lying between the points A_{m+n-2} and A_{m+n-1}. (For simplicity's sake let the k-gon be the first we are constructing in submap A. If this is not so, then the indices of the points A_i, B_i are increased.)

Now join a point P_1 between the points A_m and A_{m-1} with a point Q_2 between the points B_m and P or B_m and B_{m+1} (if $t>5$), and a point P_2 between A_m and Q or A_m and A_{m+1} (if $t>5$) with a point Q_1 between B_m and B_{m-1}. A k-gon and a t-gon and two quadrangles $P_1A_mP_2X$, $Q_1B_mQ_2X$ are created in submap A. (X is the intersection point of the lines P_2Q_1 and P_1Q_2, see Fig. 3.) Increasing the degree of the vertices P_1, P_2, Q_1, Q_2 is performed as before.

However, in forming any pair of odd-gonal faces and sometimes in forming an even-gonal face the number of vertices on a handle-opening is increased by two. Therefore the numbers of vertices on the inner handle openings must be balanced up in any step when this number is increased at one of them.

Suppose the number of vertices of an inner handle-opening O_i was increased by two. Pairs of openings on the right hand side from O_i are joined by two paths each going from one opening to the other (dot-and-dashed lines in Fig. 2) increasing the number of their vertices by two and forming new quadrangles and 4-valent vertices only. The same is done with the openings on the left hand side from O_i.

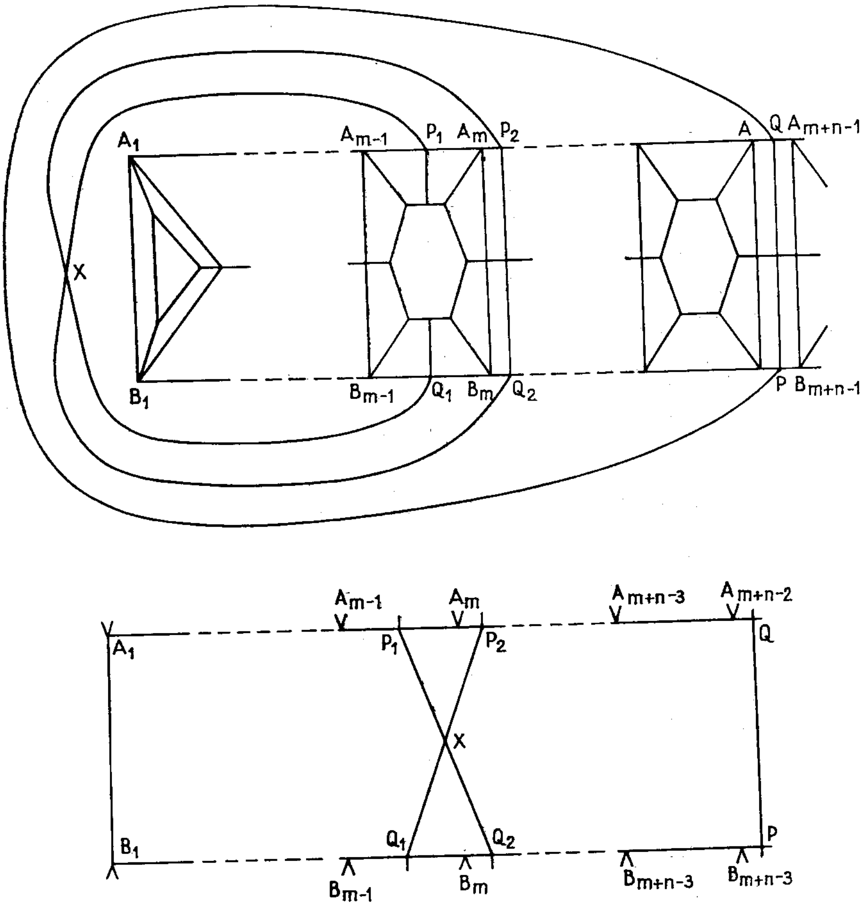

Fig. 3

In this way all the inner openings receive the same number of vertices while this need not be the case with the outer openings. However, in any case the outer openings have an odd number of vertices.

It remains, after dissecting the polygon A into r_i i-gons for all i and balancing the numbers of vertices on the inner openings to balance up the number of vertices of the outer openings when $a>b$, a or b being the numbers of vertices of these openings, respectively. Let e.g. b be the number of vertices of the opening O_1. We choose between the vertices D_1 and $E_1 (a-b)$ points $H_1, \dots, H_{a-b}$, and between the points D_2 and E_2 points $G_1, \dots, G_{a-b}$ and join these points by paths forming new quadrangles and 4-valent vertices only (see Fig. 4 where the points arised in balancing the numbers of vertices of the inner openings are not depicted.) As $g \geqq 3$ there are at least three inner openings different from O_2. Therefore we choose $\frac{1}{2}(a-b) = r$ points $L_1, \dots, L_r$ situated between Z_3 and V_3. The points $N_1, \dots, N_r$, $P_1, \dots, P_r$ and $S_1, \dots, S_r$ are likewise chosen to lie between E_4 and D_4, between Z_4 and V_4,

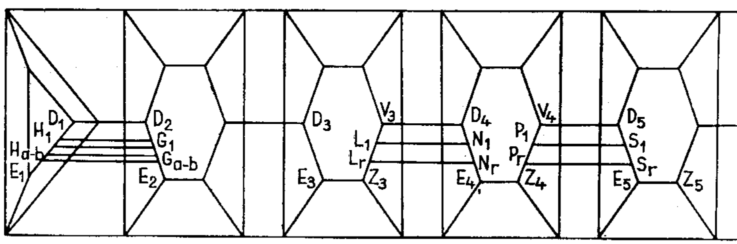

Fig. 4

and between E_5 and D_5, respectively. Then we join L_i with N_i and P_i with S_i, $i=1, \dots, r$, increasing the number of 3-valent vertices of the openings O_3 and O_5 or O_4 by r or $(a-b)$, respectively, and forming new quadrangles and 4-valent vertices. At the end of this operation the pairs of openings O_1 and O_{2g}, O_2 and O_4, O_3 and O_5 have the same number of vertices. Of course all other openings $O_6, \dots, O_{2g-1}$ have the same number of vertices too. The III. stage of the construction, the "handle forming" can follow.

β) $r_3 > 0$.

1. First let us settle the case of a sequence $p=(p_3, \dots, p_s)$ which can be decomposed into the sequence r^1 and ${}^1p'$ satisfying conditions (i), (ii) and (iv) with $r_3^1=0$, but contradicting condition (iii), i.e. $r_3=1$. In such a decomposition we should have ${}^1p_3'={}^1p_5'=1$, ${}^1p_i'=0$ for $i>5$ and the second stage of the construction could not be performed as described belove. But the realization of the sequence r^1 in submap A can be (and has been) carried out. This is done wihthout balancing up the numbers of vertices of the openings Q_1 and O_{2g}. A "wedge" (triangle) is put next to the opening O_1 (as in Fig. 5) giving a triangle and a pentagon in submap B of the constructed map. Now balancing the number of vertices of the outer openings can follow.

2. Suppose that p is not decomposable into sequences r^1 and ${}^1p'$ satisfying conditions (i), (ii) and (iv) and contradicting (iii). Then $r_3 < i-4$ for every $i>5$ for which $r_i \neq 0$. Take such an i and analogously as in α) realize in submap A the sequence $r'=(r_3', \dots, r_s')$ with $r_3'=0$, $r_i' = r_i-1$, $r_{i-r_3}' = r_{i-r_3}+1$, $r_j'=r_j$ for all $j \neq i$, $i-r_3$. One $(i-r_3)$-gon G is from the polygon A cut off as the first. The numbers of vertices of the inner handle-openings are balanced up. Then r_3 "wedges" are put next to the opening O_1 so that their vertices not lying on the opening O_1 belong to one edge of G, (see Fig. 6 where $r_3=3$). The $(i-r_3)$-gon G becomes an i-gon. Then balancing up the number of vertices of the outer openings is performed.

3. Let the sequence p be decomposable into r^1 and ${}^1p'$ satisfying (i), (ii) and (iv) and contradicting (iii), and let $r_3>1$, i.e. $r_3^1\geqq 1$, ${}^1p_3'={}^1p_5'=1$, ${}^1p_i'=0$ for $i\geqq 6$. The procedure in 1. is combined with that in 2. First the sequence r^1 is realized as in 2. Take an $i>5$ with $r_i\neq 0$ and cut off from the polygon A an $(r_i-r_3^1)$-gon, construct all remaining j-gons, $j\geqq 5$, in submap A, balance up the numbers of vertices on the inner openings, put r_3^1 "wedges" next to the openings O_1 as in Fig. 6 etc. Further a pentagon and a triangle is added in submap B as in 1. (Fig. 5).

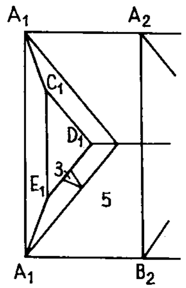

Fig. 5

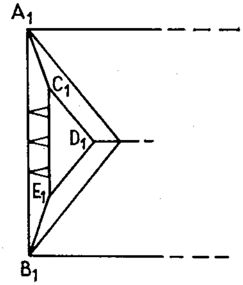

Fig. 6

II. The realization of the sequence p' on a tube is the same as in [1]. We bring this construction here for completeness' sake.

Let W be an w-gon, $w=4+\sum_{k\geqq 5}(k-4)p_k'$, with vertices $a_1, \ldots, a_w$ and right angles in the vertices a_1 and a_d, $d=\left[\frac{w+1}{2}\right]$.(See Fig. 7.)

The intersection points of the lines a_1a_2 and a_da_{d-1}, or a_wa_1 and a_da_{d+1} are the points b or c, respectively. The lines through the points a_i parallel to the line a_1c intersect the segment a_1b or a_dc in the points u_i. Analogously we obtain the points v_i on the segments a_1c, ba_d. Essential is the fact that if $\sum_{k\geqq 5}kp_k$ is even then the segments a_1b and ca_d or ca_1 and a_db contain the same number $(d-1)$ of vertices u_i or v_i. If $\sum_{k\geqq 5}kp_k$ is odd then the segments a_1b and ba_d contain $(d-1)$ points u_i

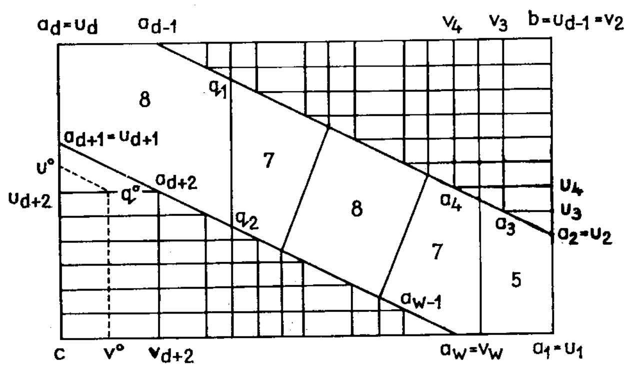

Fig. 7

or v_i, but in each of the segments $a_d c$, ca_1 there are only $(d-2)$ vertices v_i or u_i. In this case we add a new edge $u^{\cdot} q^{\cdot}$ and a series of edges on the arc $q^{\cdot} v^{\cdot}$, where the point $u^{\cdot}$ or $q^{\cdot}$ or $v^{\cdot}$ lies between the points u_{d+1} and u_{d+2} or u_{d+2} and a_{d+2} or c and v_{d+2}, respectively (dashed lines on Fig. 7). Thus all sides of the rectangle $a_1 b a_d c$ have the same number of points u_i or v_i.

Next we cut off from the polygon W the required k-gons. For some k, let $p_k' \neq 0$. We choose a point q_1 between a_n and a_{n+1}, $n<d$, and a point q_2 between a_m and a_{m+1}, $m>d$, such that $q_1 a_{n+1} a_{n+2} \dots a_d a_{d+1} \dots a_m q_2$ is a k-gon. The point q_1 is joined with a point between v_n and v_{n+1} and the point q_2 with a point between v_m and v_{m+1} by an arc.

Analogously all other required i-gons, $i \geqq 5$, are cut off from the polygon W. Along all sides of the rectangle $a_1 b a_d c$ series of quadrangles can be added. Then the trivalent vertices on one pair of opposite sides of the rectangle are unified to get a tube (or a handle).

The IIIrd stage of the construction is now simple. Pairs of the handle openings on the map R have the same number of trivalent vertices. So the handle constructed in stage II. and $(g-1)$ handles decomposed into quadrangles can be put on the map R. (The vertices on the openings of the handles are trivalent!) If the number of vertices on the opening of the tube is greater than on each handle-opening on R, the number of vertices on two handle-openings on R is increased as described in stage I. — A 2-manifold of genus g decomposed by a map realizing the sequence p is constructed.

3. THEOREM 2: *Every sequence* $p=(p_3, p_4, \dots, p_s)$ *of non-negative integers satisfying*

$$\sum_{k \geqq 3} (4-k) p_k = -8$$

for which neither $p_3 = p_{13} = 1, p_i = 0$ *for* $i \neq 3, 4, 13$, *nor* $p_5 = p_{11} = 1, p_i = 0$ *for* $i \neq 4, 5, 11$ *holds, is* 4*-realizable on the orientable surface of genus* 2.

The proof of Theorem 2 follows analogously as the proof of Theorem 1. Again the sequence p is decomposed into the sequences r and p'. However, balancing the number of vertices on the handle openings cannot be performed here as before. The cases in which this balancing should be necessary must be investigated separately. Because the number of these sequences r is relatively great we forsake to state them here. Unfortunately, we were not able either to realize the two exceptional sequences mentioned or to prove their non-realizability.

4. Remark. a) In our theorems we were interested in 4-realizing the sequence p with *any* number of quadrangles. Of course one could ask: What numbers of quadrangles can appear in these realizations? No definite answer has been produced at the time of writing.

b) The theorems 1 and 2 are analogoues of EBERHARD's theorem [2] in which first the question above was treated for 3-valent maps on the sphere. After the appearance of [3] much work was done with ramifications and analogues of EBERHARD's theorem. (See [1, 4, 5, 6, 7, 8, 9].)

Added in proof: In the meantime the sequences whose realizability is not decided by Theorem 2 turned out to be 4-realizable. (See JUCOVIČ—TRENKLER: On the structure of cell-decompositions of orientable 2-manifolds. *Mathematika* (to appear).

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